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A Technique for Relating Vehicle Structural Modes to Stiffness as Determined in Static Determinate Tests

David Griffiths, Allan Aubert, Edward R. Green, and Jim Ding
Roush Industries

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ABSTRACT

During the development of a new vehicle, the vehicle is usually tested to determine both its static torsional and bending stiffness, and its dynamic torsional and bending modes. This paper discusses a method for determining both static and dynamic properties from the modal analysis test. Such a connection between static stiffness and dynamic modes would be useful for three reasons: (1) the relative importance of apparent bending and torsion modes could be determined by their contribution to stiffness, (2) the stiffening effect of structural modifications could be determined from experimental modal tests (the modal frequency shift is also affected by any change in mass), (3) the total static compliance could easily be split on a modal basis into compliance due to the overall structure and local compliance due to local structural deflections.

INTRODUCTION

For NVH purposes vehicle structural characteristics are usually characterized either by the stiffness (or compliance) as measured in a static test, or by the structural vibration modes measured in a dynamic test. The static test is usually almost static determinate, that is the reaction forces are related to the applied forces by the test geometry. The dynamic modal tests are "free-free", that is the dynamic forces applied by the support system can be ignored. It is well known that the static compliance is important for NVH, for example low torsional compliance leads to vehicle shake, squeaks and rattles. Also low torsional compliance degrades vehicle handling. The static compliance and the modes are connected in that as stiffness is decreased (mass being held constant) the modal frequencies are decreased. For example, if a sedan is changed into a convertible by removing the greenhouse, the torsional rigidity is drastically reduced, the main torsional mode frequency drops into the range of the main suspension hop-tramp modes, and the vehicle becomes very shaky unless countermeasures are taken. This paper shows an explicit relationship between compliance measurements and dynamic modal measurements.

Conventional vehicle modal analysis is often confused by a variety of local modes. For example, seat-back fore-aft modes may also be associated with overall bending and torsion. It is not uncommon that the column bending mode is close to the body bending mode and the resultant coupling leads to difficulty in interpreting the modes. The methods described in this paper show each mode can be related to its contribution to the relevant static test, and thus how the relative importance of modes can be determined.

Frequently it would be valuable to determine the relative contribution of overall and local compliance to the total compliance. This can be done on a modal basis by separating out the contributions on the basis of frequency since the overall modes are generally at low frequencies, and the local modes contributing to total compliance occur at high frequency.

MAIN SECTION

THEORY

Given a static determinate test involving only forces and translational displacements, it can be shown (see appendix) that all the forces (F_n) can be written in terms of a single generalized force F and a set of coefficients that depend only on geometry and the scaling of the generalized force:

$$F_n = \chi_n F \tag{1}$$

$$\sum_n \chi_n = 0 \tag{2}$$

Tests involving moments and rotations can be analyzed in a similar fashion.

The static compliance (C) is calculated from the displacements d_n at the load and support points by:

$$C = \frac{\sum_n \chi_n d_n}{F} \tag{3}$$

and the contribution of a vibration mode to the compliance (c) is found from:

$$c = \frac{\Sigma_i \Sigma_j \chi_i \chi_j \Psi_i \Psi}{m(2\pi f)^2} = \frac{(\Sigma_i \chi_i \Psi_i)^2}{m(2\pi f)^2} \qquad (4)$$

where:

Ψ_i = mode shapes at load and support points

m = modal mass

f = Resonant frequency

The total compliance is the sum of the compliances due to the individual modes. The sum of the contributions of all the modes may also be found by constructing a "compliance" transfer function and extrapolating it to zero hz. The compliance function has the form:

$$\frac{d}{F} = \Sigma_i \Sigma_j \chi_i \chi_j \frac{d_i}{F_j} = -\frac{1}{(2\pi f)^2} \Sigma_i \Sigma_j \chi_i \chi_j \frac{a_i}{F_j} \qquad (5)$$

Note that the inertance function:

$$\frac{a}{F} = \Sigma_i \Sigma_j \chi_i \chi_j \frac{a_i}{F_j} \qquad (6)$$

goes to zero at zero hz and that the rigid body modes will not show in this transfer function.

BEAM BENDING COMPLIANCE

Theory

A typical beam bending application is shown in Figure 1. The beam is supported at the ends and the load is applied at the center.

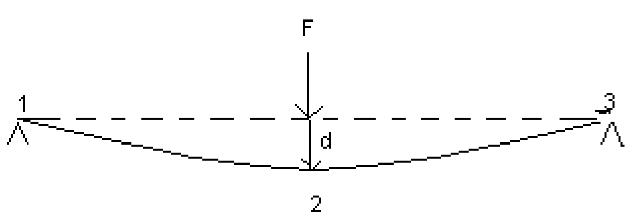


Figure 1: Beam Bending Arrangement

The natural way to scale the forces is to set:

$F_2 = F$ and $\chi_2 = 1$ (7)

The other coefficients are easily found from the conditions for static equilibrium:

$$F_1 = -\frac{F}{2} \Rightarrow \chi_1 = -\frac{1}{2} \qquad (8)$$

$$F_3 = -\frac{F}{2} \Rightarrow \chi_3 = -\frac{1}{2} \qquad (9)$$

The bending compliance is now found by substituting in the equation for compliance:

$$C_b = \frac{\sum_n \chi_n d_n}{F} = \frac{d_2 - \frac{d_1}{2} - \frac{d_3}{2}}{F} \qquad (10)$$

Note that in an ideal test d_1 and d_3 would be zero, however, even if the supports deflect the formula automatically compensates for the resulting rigid body motion.

The contribution of any mode to the compliance is then:

$$c_b = \frac{\Sigma_i \Sigma_j \chi_i \chi_j \Psi_i \Psi_j}{m(2\pi f)^2} = \frac{\left(\Psi_2 - \frac{\Psi_1}{2} - \frac{\Psi_3}{2}\right)^2}{m(2\pi f)^2} \qquad (11)$$

Comparison of Static Theory & Modal Theory

The static compliance of a simple beam as measured above is given by the standard formula:

$$C = \frac{y}{F} = \frac{L^3}{48EI} \qquad (12)$$

where:

L = Length of the Beam
E = Young's modulus of the beam material
I = Inertia of the cross-section

The free-free mode shapes of a beam are well known to have the form [1]:

$$y(x) = \frac{1}{2}\left\{\cosh(kx) + \cos(kx) - \frac{(\sinh(kL) + \sin(kL))}{[\cosh(kL) - \cos(kL)]}[\sinh(kx) + \sin(kx)]\right\} \qquad (13)$$

where we have normalized the modes shapes so that y(0)=y(L)=1, and k is a non-zero solution of:

$$\cos(kL) = \frac{1}{\cosh(kL)} \qquad (14)$$

The values of k for the n-th bending mode are given to a good approximation by:

$$k_n \cong \frac{(2n+1)\pi}{2}\frac{1}{L} \tag{15}$$

The resonant frequencies are given by:

$$f = \frac{1}{2\pi}\left[\frac{EI}{\rho}\right]^{\frac{1}{2}} k^2 \cong \frac{1}{2\pi}\left[\frac{EI}{\rho}\right]^{\frac{1}{2}}\left[\frac{(2n+1)\pi}{2L}\right]^2 \tag{16}$$

with:

ρ = Mass Per Unit Length (M/L)
M = Total Mass

The modal mass (m) is computed from the integral of the square of the mode shape and is essentially equal to a quarter of the total mass for all bending modes:

$$m = \rho\int_0^L y(x)^2 dx = \frac{\rho L}{4} = \frac{M}{4} \tag{17}$$

The first bending mode shapes at the support and load points are:

$$y(0) = y(L) = 1 \qquad y\left(\frac{L}{2}\right) = -.6078 \tag{18}$$

and substituting in the equation for modal compliance we get:

$$c_1 = \frac{(\frac{1}{2}+\frac{1}{2}+.6078)^2}{(2\pi f_1)^2 m} = \frac{L^3}{48EI} \tag{19}$$

From this we see that essentially all the compliance arises from the 1st bending mode. The second and all other even modes contribute zero because the term involving the square of mode shapes is zero. The contributions of the 3rd and 5th modes are:

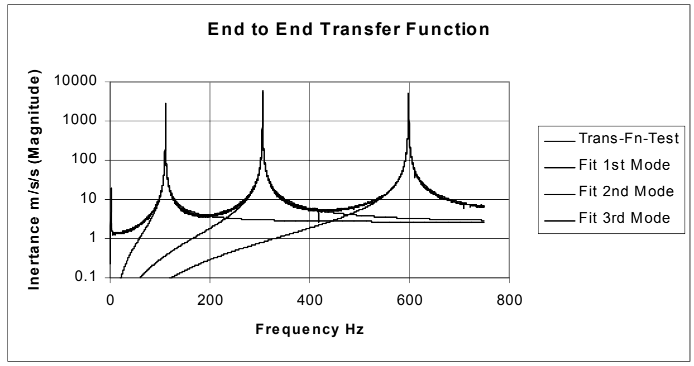
$$c_3 = \frac{1}{4352}\frac{L^3}{EI} \qquad c_5 = \frac{1}{7649}\frac{L^3}{EI} \tag{20}$$

The contribution of the 3rd mode is very low because the mode shape at the load point is positive and this causes a cancellation in the mode shape factor. The contribution of the 5th mode is low because of its high frequency. A similar pattern applies to higher modes with the result that the first mode bending dominates the compliance when the load is at the center. It is also worth noting that the static deflection shape is almost exactly the same as the first bending mode shape.

Comparison of Test, Beam Theory and CAE

To check the theory a small steel beam was subject to static and dynamic testing. The beam was also modeled using FEA, and the compliance was computed directly and from the first bending mode. The FEA model consisted of plate elements, 24 along the length of the beam and 4 across the width. The first three bending mode frequencies can be seen in the beam test data shown in figure 2 which shows the end to end transfer functions with fits to the resonant peaks. The actual values, 110.9 hz, 305.7 hz, and 589.1 hz agree well with modal theory.

Figure 2. End to End Test Transfer Functions and Fit to Resonant Peaks



The details of the beam and the results for compliance were as follows:

Beam Parameters: Length - .61 m
Width – 38.1 mm
Thickness – 7.9
Mass – 1.44 kg

Results

	Beam Theory	Test Results	FEA Results
1st Mode			
Frequency	108.5 hz	110.86	107.6 hz
Modal Mass	.36 kg	.384 kg	.366 kg
Compliance	1.55×10^{-5} m/N	1.51×10^{-5} m/N	1.55×10^{-5} m/N
Static -Comp	1.56×10^{-5} m/N	1.44×10^{-5} m/N	1.56×10^{-5} m/N

It can be seen that the various methods of estimating compliance agree to an acceptable level of accuracy.

The test results were used to create a compliance function as discussed earlier. This is shown in figure 3., together with a curve-fit to the first mode contribution.

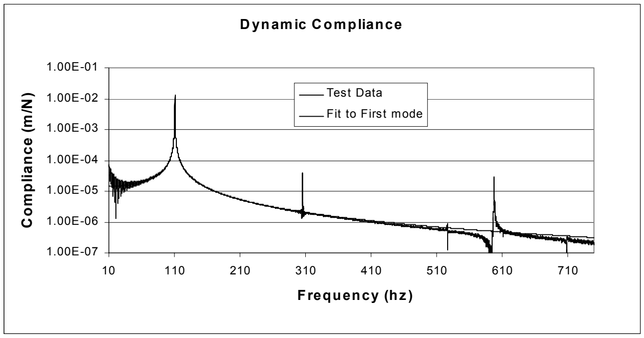


Figure 3. Dynamic Compliance Function for Beam

It can be seen that only the first bending mode shows strongly in the compliance function. The rigid body modes are "filtered out" and the function goes to the static compliance value at zero hz. The function does, however, tend to be very noisy at low frequencies due to the fact that a very wide dynamic range is required. Consequently the static value of compliance is better determined from the curve fit, which in this case gave 1.4×10^{-5} m/N.

Effect of a Local Mode – Tuned Damper

A damper tuned to the first bending mode will split the bending mode into two modes (modes 1&2) each of which has about ½ of the modal compliance. As an example, the effect of a damper attached to the center of the beam was modeled using FEA. The damper springs were set so that the damper's motion was in the direction of motion of the beam bending mode (vertical) and the degree of freedom in this direction was labeled "4". The compliance was calculated at the center of the beam and at the damper (the beam being suported at the ends). According to the theory, the compliance c_{4b} at the damper due to each mode (with bending supports) is given by:

$$c_{4b} = \frac{\Sigma_i \Sigma_j \chi_i \chi_j \Psi_i \Psi_j}{m(2\pi f)^2} = \frac{\left(\Psi_4 - \frac{\Psi_1}{2} - \frac{\Psi_3}{2}\right)^2}{m(2\pi f)^2} \tag{21}$$

and we expect that the difference of the compliance at the damper and at the center of the beam (c_{2b}) will be the reciprocal of the damper stiffness k:

$$c_{4b} - c_{2b} = \frac{1}{k} \tag{22}$$

Also, if we support the beam at the center and load at the damper, the sum of the compliances due to the modes will be $C_{42}=1/k$:

$$c_{24} = \sum_{modes} \frac{(\Psi_4 - \Psi_2)^2}{m(2\pi f)^2} = \frac{1}{k} \tag{23}$$

The model parameters and the results are shown in tables 1 & 2.

Table 1. Beam-Damper Parameters

	Res. Frequency	Mass	Compliance
Beam	107.6 hz	1.44	1.56×10^{-5} m/N (bending)
Damper	107.6 hz (grounded)	.144kg	1.52×10^{-5} m/N

Table 2a. Beam-Damper Modal Parameters

Mode #	Freq hz	ψ_1 ψ_3	ψ_2	ψ_4	Modal Mass
1	91.2	1	-0.39	-1.41	0.59
2	132.3	1	-0.834	1.617	0.89

Table 2b. Beam-Damper ModeCompliance

Mode #	Freq hz	Beam Bending ComplianceC_{2b}	Damper Bending Compliance C_{4b}	Damper Compliance C_{24}
1	91.2	1.003E-05	2.996E-05	5.322E-06
2	132.3	5.469E-06	6.189E-07	9.767E-06
	Total	1.55E-05	3.058E-05	1.509E-05

These results agree well with compliance theory.

Local Compliance -Bracket Resonances

Local compliance is frequently an issue for automotive design. For example, if a body-suspension attachment bracket is not sufficiently stiff it may be difficult to tune the suspension for vehicle dynamics and NVH. The usual rule of thumb is that the body attachment point should be at least 5, and preferably 10, times stiffer than the attached bushing or mount. This raises the questions, what do we mean by local compliance and how can we measure it? A major issue is to separate the local compliance from the global compliance. To demonstrate how this can be done using the dynamic compliance method, we consider an attachment bracket at the center of the beam using a mass spring system tuned so that the bracket resonant frequency is 20% above the bending frequency (when the bracket is grounded). The mode shapes and compliances obtained from CAE are given in tables 3 and 4:

Table 3. Beam-Bracket Parameters

	Res. Frequency	*Mass*	*Compliance*
Beam	*107.6 hz*	*1.44*	*1.56×10^{-5} m/N (bending)*
Bracket	*129 hz (grounded)*	*.144kg*	*1.06×10^{-5} m/N*

Table 4a. Beam-Bracket Modes

Mode #	Freq hz	ψ_1 ψ_3	ψ_2	ψ_4	Modal Mass
1	95.5	1	-0.4568	-1.01	0.462
2	151.4	1	-0.9384	2.49	1.517

Table 4b. Beam-Bracket Modal Compliance

Mode #	Freq hz	Beam Bending ComplianceC_{2b}	Bracket-	Bracket

			Bending Compliance C_{4b}	Compliance C_{24}
1	95.5	1.275E-05	2.433E-05	1.84E-06
2	151.4	2.738E-06	1.620E-06	8.57E-06
	Totals	1.549E-05	2.592E-05	1.04E-05

From a CAE point of view the local compliance can be determined relatively easily by applying the appropriate loads and constraints in a simulation of a static test. Fixturing issues and accessibility will often make a physical test difficult, and in that case the dynamic compliance approach has considerable merit since it can be implemented using accelerometers and an impact hammer.

Multiple Local Resonances

Determining the “real” bending mode may be difficult when there are multiple local resonances. This is illustrated below for a beam with two “brackets” attached to its center.

Table 5. Parameters for Multiple Local Resonance

	Res. Frequency	*Mass*	*Compliance*
Beam	*107.6 hz*	*1.44*	1.56×10^{-5} m/N (bending)
Bracket (a)	*86 hz*	*.144kg*	*2.38×10^{-5} m/N*
Bracket (b)	*129 hz*	*.144kg*	*1.06×10^{-5} m/N*

Table 6. Modal Compliance Results

Mode #	Freq hz	ψ_1 ψ_3	ψ_2	Modal Mass	Beam Bending ComplianceC_{2b}
1	81.0	-1.035	0.242	1	6.295E-06
2	103.5	1.038	-0.582	1	6.206E-06
3	154.5	0.860	-0.818	1	2.988E-06
				Sum	1.549E-05

Here we see that the bending compliance has been spread over a 70 hz range, and even the highest mode has 20% of the compliance. The underlying bending mode frequency can be roughly estimated by weighting the frequencies by the bending compliance. This estimate tends to be a little low, in this case 104 hz. A simple linear average tends to be slightly high (113 hz).

Simulated transfer functions show the problem very clearly. Figure 4. below shows a simulation of the beam center point driving point function with 3 added local resonances. The first mode bending is, to say the least, hard to recognize.

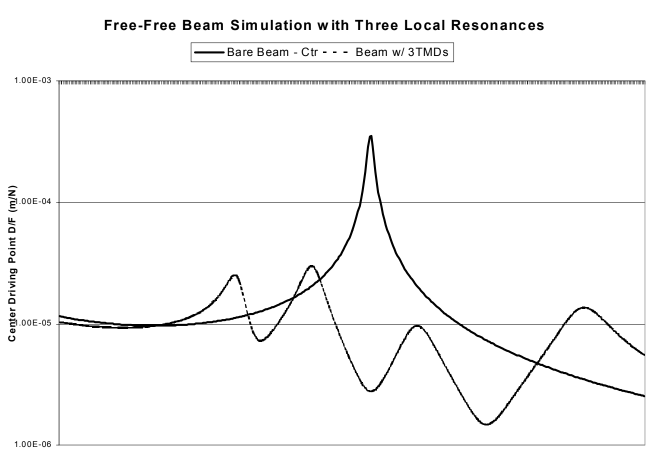


Figure 4. Center Point Driving Point of a Beam with 3 Local Resonances

The implication for automotive applications is that local resonant modes of trim items will split major bending and torsion modes into a series of modes each of which has part of the overall compliance. The frequency of the underlying mode could be roughly estimated by averaging the frequencies of the sub-modes weighted by their contribution to the compliance. The key issue is, however, will high bending or torsional compliance result in a shaky vehicle. One obvious approach is to set not-to-exceed target curves for dynamic compliance, and to associate them with targets for static compliance.

FRAME TORSIONAL COMPLIANCE

Theory

A typical frame torsion application is shown in figure 5. The load is applied at points 1 and 4, and the frame is supported at points 2 and 3.

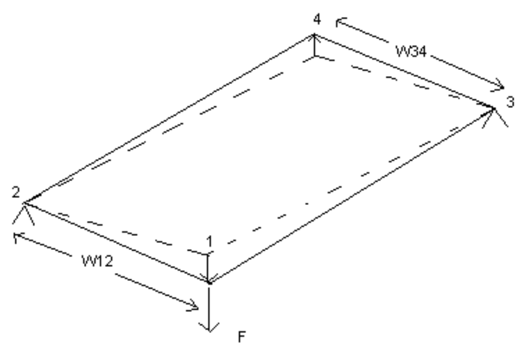


Figure 5: Frame Torsion

The motion of the frame combines both frame torsion and a net rigid body motion that must be removed in the calculation of the torsional compliance.

In this case the generalized force is the applied torque, Γ, and the applied forces and reactions are defined by:

$$F_i = \chi_i \Gamma \qquad (24)$$

$$\chi_1 = \frac{1}{W_{12}} = -\chi_2 \qquad \chi_3 = -\frac{1}{W34} = -\chi_4 \qquad (25)$$

Assuming small angles, the net twist angle (radians) in the frame is: test

$$\Omega = \frac{d_1 - d_2}{W_{12}} - \frac{d_3 - d_4}{W_{34}} = \sum_i \chi_i d_i \qquad (26)$$

The torsional compliance in radians per Nm is then:

$$T = \frac{\Omega}{\Gamma} = \frac{\sum_i \chi_i d_i}{\Gamma} \tag{27}$$

The contribution of an individual mode to the torsional compliance can now be written:

$$\tau = \frac{\sum_i \sum_j \chi_i \chi_j \Psi_i \Psi_j}{m(2\pi f)^2} = \frac{\left(\sum_i \chi_i U_i\right)^{2\ 2}}{m(2\pi f)^2}$$

$$= \frac{1}{m(2\pi f)^2}\left(\frac{\Psi_1 - \Psi_2}{W_{12}} - \frac{\Psi_3 - \Psi_4}{W_{34}}\right) \tag{28}$$

Test Frame Characteristics

A modified truck frame was selected for a test comparison (the frame had been shortened for earlier hardware studies, and it was necessary to weld on a new front cross-member). The frame had the following characteristics:

Frame Mass = 135 kg   Width W_{12} = .972 m
Width W_{34} = .946 m

Static Torsional Test

The frame was supported at three corners and loaded at the 4th corner. Based on the load-deflection curve the torsional rigidity (static) was determined to be:

$$T = \frac{\Omega}{\Gamma} = \frac{\sum_i \chi_i d_i}{\Gamma} = 1.61x10^{-5}\, rad / N.m \tag{29}$$

Dynamic Torsional Test – 1st Torsional Mode

The frame was supported free-free and impact testing was used to determine the transfer functions between the four corners used for the static test. Table 6. shows the frequency of the first torsional mode along with the mode shapes at the corners, the modal mass and the damping (hysterectic factor η)

Table 6.a Torsional Mode Parameters

Frequency	Ψ_1	Ψ_2	Ψ_3	Ψ_4
25.7 hz	1	-.975	-.898	.942

Table 6.b Torsional Mode Parameters (cont)

Modal Mass	Damping (η)	Torsional Compliance
41.9 kg	.018	1.44 x 10^{-5} rad/N.m

Dynamic Torsional Test – Torsional Compliance Function

A torsional compliance function was created using the previously described method and the 16 transfer functions between the corners. The resulting function is shown below together with a fit to the first mode contribution.

The compliance function makes it clear that in this case the lowest torsional mode dominates the torsional compliance. The fit value of the compliance function was 1.58x10-5 rad/N.m in good agreement with the static test results.

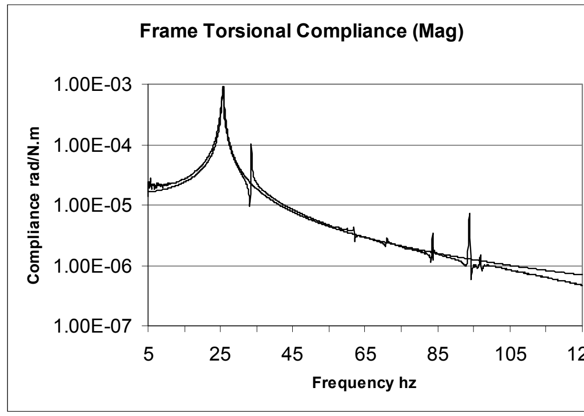


Figure 6. Frame Torsional Compliance

CONCLUSION

The results shown in this paper demonstrate the relationship between dynamic structural modes and static stiffness as determined in the usual laboratory tests. These relationships are useful in a number of ways including:

- determining the relative importance of apparent bending and torsion modes to static stiffness
- determining the stiffening effect of structural modifications using experimental modal tests
- determining how the total static compliance depends on overall structure and local compliance due to local structural deflections

REFERENCES

Arnold Sommerfeld, *Partial Differential Equations in Physics*, Academic Press, 1949, pp. 303-304

APPENDIX

Derivation of Formulas

Reaction Forces in Static Determinate Test – Generalized Force

Consider a static determinate test in which the structure is supported by the minimum set of supports. If an infinitesimal displacement d_s is made at one of the support locations, then the displacement d_i at a load location i will be given by:

$$d_i = -\beta_{is} d_s \tag{30}$$

where β_{is} is a coefficient depending only on the geometry of the test. From the principle of virtual work [2] the total work W done by the load and reaction forces is zero so:

$$W = 0 = F_s d_s + \sum_i F_i d_i = F_s d_s - \sum_i \beta_{is} F_i d_s \tag{31}$$

and from this we can deduce that the relationship between the loads and the reaction forces is:

$$F_s = \sum_i \beta_{is} F_i \tag{32}$$

If we scale the input forces to a single generalized force F:

$$F_i = \alpha_i F \tag{33}$$

then the reaction forces become:

$$F_s = \sum_i \beta_{is} \alpha_i F = \chi_s F \tag{34}$$

where $$\chi_s = \sum_i \beta_{is} \alpha_i \tag{35}$$

Further, we can equate χ and α for the applied loads and write all the applied loads and reaction forces in the form:

$$F_j = \chi_j F \tag{36}$$

Definition of Static Compliance

Static Compliance is defined to be the ratio of the generalized displacement D to the generalized force F. It is natural to define the generalized displacement with the same χ's as define the applied load and the reaction forces:

$$D = \sum_j \chi_j y_j \tag{37}$$

y_j=displacements at load and support points

This definition automatically removes any rigid body deflections at the load points that result from the deflection of the supports. The definition of static compliance now becomes:

$$C = \frac{D}{F} = \frac{\sum_j \chi_j y_j}{F} \tag{38}$$

Static Compliance and Vibration Modes

The contribution of individual modes is additive, so for simplicity of notation we will only consider a single vibration mode. The displacement y_i due a mode depends on its modal participation factor γ and the mode shape ψ_i via [3]:

$$y_i = \Psi_i \gamma \tag{39}$$

The equation of motion for γ is:

$$m\ddot{\gamma} + c\dot{\gamma} + k\gamma = \sum_j \Psi_j F_j \tag{40}$$

where: m = modal mass c=viscous damping factor

As a consequence modal participation factor at zero hz is:

$$\gamma = \frac{\sum_j \Psi_j F_j}{k} \tag{41}$$

and the static displacement due to a mode is:

$$y_i = \frac{\Psi_i \sum_j \Psi_j F_j}{k} \tag{42}$$

Inserting this into the equation for compliance yields:

$$C = \frac{\sum_i \chi_i y_i}{Fk} = \frac{\sum_i \chi_i \Psi_i \sum_j \Psi_j \chi_j F}{Fk} =$$

$$\frac{\sum_{i,j} \chi_i \chi_j \Psi_i \Psi_j}{k} = \frac{\left[\sum_i \chi_i \Psi_i\right]^2}{(2\pi f)^2 m} \tag{43}$$

Since this is a quadratic form the compliance is positive as long as the mode shapes are real (normal).